ON COMPOSITE FORMAL POWER SERIES

JACQUES CHAUMAT AND ANNE-MARIE CHOLLET

ABSTRACT. Let F be a holomorphic map from $\mathbb{C}^n$ to $\mathbb{C}^n$ defined in a neighborhood of 0 such that $F(0) = 0$. If the Jacobian determinant of F is not identically zero, P. M. Eakin et G. A. Harris proved the following result: any formal power series such that $\mathcal{A} \circ F$ is analytic is itself analytic. If the Jacobian determinant of F is identically zero, they proved that the previous conclusion is no more true.

The authors get similar results in the case of formal power series satifying growth conditions, of Gevrey type for instance. Moreover, the proofs here give, in the analytic case, a control of the radius of convergence of $\mathcal{A}$ by the radius of convergence of $\mathcal{A} \circ F$.

RÉSUMÉ. Soit F une application holomorphe de $\mathbb{C}^n$ dans $\mathbb{C}^n$ définie dans un voisinage de 0 et vérifiant $F(0) = 0$. Si le jacobien de F n'est pas identiquement nul au voisinage de 0, P.M. Eakin et G.A. Harris ont établi le résultat suivant: toute série formelle $\mathcal{A}$ telle que $\mathcal{A} \circ F$ est analytique est elle-même analytique. Si le jacobien de F est identiquement nul, ils montrent que la conclusion précédente est fausse.

Les auteurs obtiennent des résultats analogues pour les séries formelles à croissance contrôlée, du type Gevrey par exemple. De plus, les preuves données ici permettent, dans le cas analytique, un contrôle du rayon de convergence de $\mathcal{A}$ par celui de $\mathcal{A} \circ F$.

INTRODUCTION

We denote by $\mathcal{F}_n$ the ring of formal power series in n variables. Let $\mathcal{A} \in \mathcal{F}_n$; we write

$$\mathcal{A} = \sum_{J \in \mathbb{N}^n} a_J x^J = \sum_{j \in \mathbb{N}} \sum_{J \in \mathbb{N}^n; |J| = j} a_J x^J = \sum_{j \in \mathbb{N}} H_j \mathcal{A}(x),$$

where $| J |$ denotes the length of the multi-index J; $H_j \mathcal{A}(x)$ is the homogeneous polynomial of degree j in the expansion of $\mathcal{A}$. We shall use the same notation $H_j \Phi(x)$ in the power series expansion of a holomorphic function Φ in a neighborhood of 0.

Let $\mathrm{ord}(\mathcal{A}) = \inf(j \in \mathbb{N}; H_j \mathcal{A}(x) \not\equiv 0)$ be the vanishing order of $\mathcal{A}$, with the convention $\mathrm{ord}(\mathcal{A}) = \infty$ if $\mathcal{A}$ is identically 0. The same notation will also be used for the vanishing order of a holomorphic function in a neighborhood of 0.

Let F be a holomorphic map from $\mathbb{C}^n$ to $\mathbb{C}^n$ defined in a ball $B(0, r_0)$ of $\mathbb{C}^n$ and such that $F(0) = 0$. We denote by Φ the Jacobian determinant of F.

Theorem A [2], [4], [7]. *The following properties are equivalent*:

(A.1) $$\mathrm{ord}(\Phi) < \infty,$$

(A.2) *there exist constants D_1 and D_2 such that*
for any $\mathcal{A} \in \mathcal{F}_n$, $\mathrm{ord}(\mathcal{A} \circ F) \leq D_1 \mathrm{ord}(\mathcal{A}) + D_2$.

D_1 and D_2 depend only on F.

Condition (A.1) implies (A.2) is a step in the proof of the Glaeser Theorem on Composite Functions [4], [7]. We can deduce from it the estimate $D_1 \leq 2\mathrm{ord}(\Phi) - 1$. The converse is proved by P. M. Eakin and G. A. Harris in ([2], lemma 4.2, inequality 4.2.2).

We denote by $\mathcal{O}(n)$ the ring of convergent power series in n variables.

Theorem B [2]. *The following properties are equivalent*:

(B.1) $$\mathrm{ord}(\Phi) < \infty,$$

(B.2) *for any $\mathcal{A} \in \mathcal{F}_n$, $\mathcal{A} \circ F \in \mathcal{O}(n)$ implies $\mathcal{A} \in \mathcal{O}(n)$.*

As mentioned by P. M. Eakin and G. A. Harris, this result can be seen as a corollary of a more general result of A. M. Gabrielov [3].

In this work, the authors deal with formal power series whose coefficients satisfy growth conditions. More precisely, let $M = \{M_p\}_{p \geq 0}$ be a sequence of real strictly positive numbers satisfying the following properties:

(H_1) $M_0 = 1$ and $\{M_p\}_{p \geq 0}$ is logarithmically convex,
(H_2) $M_p^{1/p}$ tends to infinity with p.

Let C be a positive constant and, for any $\mathcal{A}$ of $\mathcal{F}(n)$, let

$$\| \mathcal{A} \|_{C,M} = \sup_{J \in \mathbb{N}^n} \frac{|a_J|}{C^j M_j}.$$

Here j denotes the length of J.

Let d be a strictly positive integer. We denote by $M^{(d)}$ the sequence defined by

$$M_n^{(d)} = M_{dn}, \text{ for any integer } n.$$

Theorem C. *Let F be a holomorphic map from $\mathbb{C}^n$ to $\mathbb{C}^n$ defined in a neigborhood of 0 and let Φ be its Jacobian determinant. Let M be a sequence of strictly positive real numbers satisfying (H_1) and (H_2). Let C be a strictly positive constant. The following properties are equivalent:*

(C.1) $$\mathrm{ord}(\Phi) < \infty,$$

(C.2) *there exist $D > 0$ and an integer $d \geq 1$ such that*
for any $\mathcal{A} \in \mathcal{F}_n$, $\|\mathcal{A}\|_{D,M^{(d)}} \leq D\|\mathcal{A} \circ F\|_{C,M}$.

The techniques developed here allow us to get precise norm estimates. The parameter d which occurs in (C.2) is optimal, as shown in Remark 18 (a). Moreover, we also get the best constants one should expect in (A.2), as shown in Theorem 17.

We can also observe that the assumptions on the sequence M are extremely weak. If $M_p = p!^\alpha, \alpha > 0$, we get the classical Gevrey growth condition. It can be verified that the method developed here can be adapted if the sequence M_p is identically equal to 1. This gives a new proof that (B.1) implies (B.2); as in [8],

we then get a control of the radius of convergence of $\mathcal{A}$ in terms of the radius of convergence of $\mathcal{A} \circ F$.

1. Let F be a holomorphic map from $\mathbb{C}^n$ to $\mathbb{C}^n$ defined in a ball $B(0, r_0)$ in $\mathbb{C}^n$ such that $F(0) = 0$. Let g be a holomorphic function in a neighborhood of 0. For any multi-index L in $\mathbb{N}^n$ of length $\ell \geq 1$, we have in a neighborhood of 0

$$(1.1) \qquad \frac{\partial^\ell}{\partial u^L}(g \circ F)(x) = \sum_{K \in \mathbb{N}^n; 1 \leq |K| \leq \ell} A_{K,L}(x) \frac{\partial^k}{\partial z^K}(g) \circ F(x)$$

with $A_{K,L}$ holomorphic functions in $B(0, r_0)$.

Of course, this formula remains true if g is a formal power series. Moreover, we have the following lemma, which can be proved by induction as in [6, proposition 2.5].

Lemma. *There exists a constant $C_1 > 0$, depending only on F, such that, for all multi-indices L, K and S and any x in $B(0, \frac{r_0}{2})$,*

$$(1.2) \qquad \left| \frac{\partial^s}{\partial x^S} A_{K,L}(x) \right| \leq C_1^{\ell+s+1}(\ell + s - k)!.$$

Here, k, ℓ and s denote the lengths of K, L and S.

Hence, we deduce immediately that there exists a constant $C_2 > 0$, depending only on F, such that, for any $\mathcal{A} \in \mathcal{F}_n$ and any $C > 0$,

$$(1.3) \qquad \| \mathcal{A} \circ F \|_{C_2 C, M} \leq C_2 \| \mathcal{A} \|_{C,M} .$$

Our purpose here is to obtain a converse. In fact, we want to discuss the following question:

> *If, for some $C, \| \mathcal{A} \circ F \|_{C,M}$ is finite, what can be said of $\mathcal{A}$?*

Inequality (1.3) gives an evident answer if F is a biholomorphic map in a neighborhood of 0.

In what follows we assume that the Jacobian determinant Φ of F satisfies

$$(1.4) \qquad \Phi(0) = 0 \quad \text{and} \quad \mathrm{ord}(\Phi) < \infty.$$

Write

$$(1.5) \qquad \Phi(x) = \sum_{j \in \mathbb{N}} \sum_{J \in \mathbb{N}^n; |J|=j} \varphi_J x^J = \sum_{j \in \mathbb{N}} H_j \Phi(x)$$

and $\mu = \mathrm{ord}(\Phi)$; we get then

$$(1.6) \qquad \begin{aligned} H_j \Phi(x) &\equiv 0 \quad \text{if } j < \mu, \\ H_\mu \Phi(x) &\not\equiv 0. \end{aligned}$$

Let

$$(1.7) \qquad \Sigma = \Phi^{-1}(0).$$

2. Geometrical Lemma. *There exists an open subset Ω_1 of the unit sphere in $\mathbb{C}^n$ such that*:

$$(2.1) \qquad z e^{i\theta} \in \Omega_1, \ \textit{if } z \textit{ belongs to } \Omega_1 \textit{ and } \theta \textit{ to } [0, 2\pi].$$

$$(2.2) \qquad \begin{aligned} &\textit{There exists } C_3 > 0 \textit{ such that, for any } z \textit{ of } \Omega_1, \\ &\qquad |H_\mu \Phi(z)| \geq C_3. \end{aligned}$$

(2.3)
$$\text{There exists } r_1 > 0 \text{ such that,}$$
$$\text{for any } z \text{ in } \Omega_1 \text{ and any } \zeta \text{ in } \mathbb{C}, \mid \zeta \mid < r_1,$$
$$\mid \Phi(\zeta z) \mid \geq \tfrac{C_3}{2} \mid \zeta \mid^\mu .$$

(2.4)
$$\text{There exists } C_4 > 0 \text{ such that,}$$
$$\text{for any } z \text{ in } \Omega_1 \text{ and any } \zeta \text{ in } \mathbb{C}, \mid \zeta \mid < r_1,$$
$$C_4 \mid \zeta \mid \leq \text{dist}(\zeta z, \Sigma) \leq \mid \zeta \mid .$$

Proof. This is a simple consequence of the definition of μ and of the holomorphy of Φ.

This open subset Ω_1 will play an important role in the definition of a new family of norms on $\mathcal{F}_n$ (see 7.1).

3. Definitions and Notations. Let Γ_1 be the cone defined by

(3.1)
$$\Gamma_1 = \{\zeta z; z \in \Omega_1, \zeta \in \mathbb{C}, \mid \zeta \mid < r_1\}.$$

Clearly, $\Gamma_1 \cap \Sigma = \{0\}$.

Let $x = \{x_1, \cdots, x_n\}$ be a point of $B(0, r_1)\backslash\Sigma$ and g a holomorphic function in a neighborhood of $F(x)$. We have

(3.2)
$$\frac{\partial}{\partial z_p}(g \circ F)(x) = \sum_{q=1}^{n} \frac{\partial}{\partial u_q}(g) \circ F(x)\frac{\partial}{\partial z_p}(F_q)(x), \qquad 1 \leq p \leq n.$$

As, by hypothesis, the Jacobian matrix JF of F is invertible at the point x, we get

(3.3)
$$\frac{\partial}{\partial u_q}(g) \circ F(x) = \frac{1}{\Phi(x)} \sum_{p=1}^{n} T_{p,q}(x)\frac{\partial}{\partial z_p}(g \circ F)(x), \qquad 1 \leq q \leq n.$$

The functions $T_{p,q}$, $1 \leq p \leq n, 1 \leq q \leq n$, are holomorphic in the ball $B(0, r_1)$. Let ν be the minimum of vanishing orders of the functions $T_{p,q}$. It is the vanishing order of the matrix of cofactors of $JF(x)$. Then, there exists a constant $C_5 > 0$ such that, for any $p, 1 \leq p \leq n$, any $q, 1 \leq q \leq n$, and any x in $B(0, r_1)$,

(3.4)
$$\mid T_{p,q}(x) \mid \leq C_5 \mid x \mid^\nu .$$

It follows from (1.4) that, for any x in $B(0, r_1)$, with a suitable constant C_6, we have

(3.5)
$$\mid \Phi(x) \mid \leq C_6 \mid x \mid^\mu \text{ and hence } \mu \geq \nu.$$

Observe that

(3.6)
$$\mu > \nu.$$

Indeed, for any z in Ω_1 and any ζ such that $\mid \zeta \mid < r_1$, we get, from (2.3) and (3.5), $\frac{C_3}{2} \mid \zeta \mid^\mu \leq \mid \Phi(\zeta z \mid \leq C_6 \mid \zeta \mid^\mu$. If $\mu = \nu$, it follows that $JF^{-1}(\zeta z)$ is bounded. Then, letting ζ tend to 0, we get that $JF(0)$ is invertible. This contradicts (1.4).

Replace g in (3.3) by $\frac{\partial}{\partial u_m}(g), 1 \leq m \leq n$. We get then an expression of the derivatives of order 2 of g evaluated at $F(x)$ in terms of the derivatives of order ≤ 2 of $g \circ F$ evaluated at x. By iterating the process, we get, for any multi-index L of $\mathbb{N}^n$ of length $\ell \geq 1$ and any x in $B(0, r_1)\backslash\Sigma$,

(3.7)
$$\frac{\partial^\ell}{\partial u^L}(g) \circ F(x) = \frac{1}{\Phi(x)^{2\ell-1}} \sum_{K \in \mathbb{N}^n; 1 \leq |K| \leq \ell} T_{K,L}(x)\frac{\partial^k}{\partial z^K}(g \circ F)(x)$$

with $T_{K,L}$ holomorphic functions in $B(0, r_1)$.

4. Lemma. *For any z in Γ_1 and any ζ in $\mathbb{C}$, $| \zeta | \leq r_1$, the functions $T_{K,L}$ appearing in (3.7) satisfy the following properties:*

$$(4.1) \qquad \begin{aligned} &T_{K,L}(z\zeta) = \zeta^{k-\mu+\ell(\mu+\nu-1)} \sum_{j=0}^{\infty} C_{K,L,j}(z)\zeta^j, \\ &| C_{K,L,j}(z) | \leq r_1^{-j} C_7^{\ell+1}(\ell-k)!. \end{aligned}$$

Proof. Of course, $k - \mu + \ell(\mu + \nu - 1) \geq 0$.

We use (2.3) and (3.4) and we apply proposition 2.5 of [6] with $\omega(t) = t^{\mu-\nu+1}$ and $E = \Gamma_1$. Hence, there exists (after shrinking r_1 and Ω_1 if necessary) a constant $C_8 \geq 1$ such that, for any z in Ω_1 and any ζ, $0 < | \zeta | < r_1$,

$$(4.2) \qquad \left| \frac{T_{K,L}}{\Phi^{2\ell-1}}(\zeta z) \right| \leq C_8^{\ell+1}(\ell-k)! \, | \zeta |^{k-\ell(\mu-\nu+1)} .$$

From (4.2) and (3.5), for any z in Ω_1 and any ζ, $0 < | \zeta | < r_1$, we have

$$(4.3) \qquad |T_{K,L}(\zeta z)| \leq C_9^{\ell+1}(\ell-k)! \, | \zeta |^{k-\mu+\ell(\mu+\nu-1)} .$$

with C_9 a suitable constant.

Let $D(0, r_1)$ denote the disk of center 0 and radius r_1. It follows from (4.3) that, for any given z in Ω_1, the function

$$\zeta \to \frac{T_{K,L}(\zeta z)}{\zeta^{k-\mu+\ell(\mu+\nu-1)}}$$

is holomorphic and bounded in $D(0, r_1)\backslash\{0\}$. It extends to a function holomorphic in $D(0, r_1)$ and bounded by $C_9^{k+1}(\ell-k)!$. It has a power series expansion in $D(0, r_1)$; this proves (4.1) by using Cauchy inequalities.

5. Lemma.

$$(5.1) \qquad \begin{aligned} H_j T_{K,L} &\equiv 0 \;\; \text{if } j < k - \mu + \ell(\mu + \nu - 1) \\ &\text{and, for any } z \text{ in } \Omega_1, \end{aligned}$$

$$| H_j T_{K,L}(z) | \leq C_{10}^{\ell+1}(\ell-k)! r_1^{k-\mu+\ell(\mu+\nu-1)-j} \;\; \text{if } j \geq k - \mu + \ell(\mu + \nu - 1)$$

with C_{10} a suitable constant.

Proof. This is just an other way of stating lemma 4.

6. Remark. From (3.7) we have, for any x in $B(0, r_1)$ and any multi-index L in $\mathbb{N}^n$ of length ℓ,

$$(6.1) \qquad \Phi(x)^{2\ell-1} \frac{\partial^\ell}{\partial u^L}(g) \circ F(x) = \sum_{K\in\mathbb{N}^n;|K|\leq\ell} T_{K,L}(x)\frac{\partial^k}{\partial z^K}(g \circ F)(x).$$

This formula still makes sense if we replace g by a formal power series $\mathcal{A}$. We have the equality (6.2) in the ring $\mathcal{F}_n$:

$$(6.2) \qquad \Phi^{2\ell-1} \frac{\partial^\ell}{\partial u^L}(\mathcal{A}) \circ F = \sum_{K\in\mathbb{N}^n;|K|\leq\ell} T_{K,L}\frac{\partial^k}{\partial z^K}(\mathcal{A} \circ F).$$

7. Definition. We introduce now a new family of norms on $\mathcal{F}_n$ well adapted to Ω_1 and to the estimates obtained in lemma 5. Denote, for any $\mathcal{A}$ in $\mathcal{F}_n$,

$$(7.1) \qquad \| \mathcal{A} \|_{C,M,\Omega_1} = \sup_{j\in\mathbb{N}} \frac{\displaystyle\sup_{z\in\Omega_1} | H_j\mathcal{A}(x) |}{C^j M_j}.$$

8. Key Lemma. *There exist two strictly positive constants C_{11} and C_{12}, depending only on Ω_1 such that, for any $\mathcal{A}$ in $\mathcal{F}_n$ and any $C > 0$,*

$$(8.1) \qquad \| \mathcal{A} \|_{C,M} \leq \| \mathcal{A} \|_{C_{11}C,M,\Omega_1} \leq \| \mathcal{A} \|_{C_{12}C,M} .$$

We have, for any $\mathcal{A}$ in $\mathcal{F}_n$, any constant C and any multi-index K of length k,

$$(8.2) \qquad \sup_{|x|=1} \left| H_j \frac{\partial^k}{\partial z^K}(\mathcal{A})(x) \right| \leq k!(2nC)^{j+k} M_{j+k} \| \mathcal{A} \|_{C,M} .$$

Proof. It is known from [1] that there exists a constant $C_{13} > 0$ such that, for any homogeneous polynomial of degree j in the variable x in $\mathbb{C}^n$, we have

$$(8.3) \qquad C_{13}^j \sup_{J \in \mathbb{N}^n} | p_J | \leq \sup_{|x|=1} | P_j(x) | \leq n^j \sup_{J \in \mathbb{N}^n} | p_J | .$$

Here we write $P_j(x) = \sum_{J \in \mathbb{N}^n; |J|=j} p_J x^J$.

It is known also from [5] that, if Ω_1 is an open subset of the unit sphere in $\mathbb{C}^n$, there exists a constant $C_{14}(\Omega_1)$ such that, for any homogeneous polynomial P_j of degree j, we have

$$(8.4) \qquad \sup_{|x|=1} | P_j(x) | \leq (C_{14}(\Omega_1))^j \sup_{x \in \Omega_1} | P_j(x) | .$$

From (8.3) and (8.4), we get (8.1).

In order to prove (8.2), we write

$$(8.5) \qquad \frac{\partial^k}{\partial z^K}(\mathcal{A}) = \sum_{T \geq K} a_T \frac{\partial^k}{\partial z^K}(x^T) = \sum_{T \geq K} a_T C_{T,K} x^{T-K}$$
$$\text{with } | C_{T,K} | \leq k! 2^{|T|}.$$

Inequality (8.2) is an immediate consequence of (8.5) and (8.3).

9. Definition. To the sequence $M = \{M_p\}_{p \geq 0}$, we associate the function h_M defined by

$$(9.1) \qquad h_M(r) = \inf_{p \geq 0} r^p M_p, \ r > 0.$$

From (H_2), we claim that h_M is a strictly positive increasing function.

10. Lemma. *For any $\mathcal{A}$ in $\mathcal{F}_n$ and any multi-index L in $\mathbb{N}^n$ of length ℓ, put*

$$(10.1) \qquad \mathcal{S}_L = \sum_{K \in \mathbb{N}^n; |K| \leq \ell} T_{K,L} \frac{\partial^k}{\partial z^K}(\mathcal{A} \circ F).$$

Then

(10.2)

$H_j \mathcal{S}_L \equiv 0 \ for \ j \leq \ell(\mu + \nu - 1) - \mu$ and

$\sup_{z \in \Omega_1} | H_j \mathcal{S}_L(z) |$

$\qquad \leq h_M^{-1}(C_{15}C) \| \mathcal{A} \circ F \|_{C,M} \ \ell! C_{16}^{\ell+1}(C_{16}C)^{j+\mu-\ell(\mu+\nu-1)} M_{j+\mu-\ell(\mu+\nu-1)}$

$\qquad\qquad\qquad\qquad\qquad\qquad if \ j > \ell(\mu + \nu - 1) - \mu.$

Here, C_{15} and C_{16} are suitable strictly positive constants depending only on F.

Proof. Write, for simplicity,

$$(10.3) \qquad \mathcal{B}_K = \frac{\partial^k}{\partial z^K}(\mathcal{A} \circ F) = \sum_j H_j \mathcal{B}_K(x).$$

We get

$$(10.4) \qquad T_{K,L}\frac{\partial^k}{\partial z^K}(\mathcal{A} \circ F) = \sum_{m \in \mathbb{N}} \sum_{q+j=m} H_q T_{K,L}(x) \ H_j \mathcal{B}_K(x).$$

Using then (5.1), (7.1) and (8.2), we get

(10.5)

$$H_m\left(T_{K,L} \ \mathcal{B}_K\right) \equiv 0, \text{ for } m < k - \mu + \ell(\mu + \nu - 1)$$

and, for $m \geq k - \mu + \ell(\mu + \nu - 1)$ and z in Ω_1,

$$|H_m\left(T_{K,L} \ \mathcal{B}_K\right)| \leq \sum_{q+j=m; q \geq k-\mu+\ell(\mu+\nu-1)} C_{10}^{\ell+1}(\ell-k)! r_1^{k-\mu+\ell(\mu+\nu-1)-q}$$

$$\times (2nC)^{j+k} M_{j+k} k! \parallel \mathcal{A} \circ F \parallel_{C,M}$$

$$\leq \ell! C_{10}^{\ell+1} \parallel \mathcal{A} \circ F \parallel_{C,M} \sum_{q+j=m; q \geq k-\mu+\ell(\mu+\nu-1)} r_1^{k-\mu+\ell(\mu+\nu-1)-q} (2nC)^{j+k} M_{j+k}.$$

Observe now, from definition (9.1), that, for any $s \in \mathbb{N}, s > 0$,

$$(10.6) \qquad r_1^{-s} \leq (nC)^s M_s h_M^{-1}\left(r_1 nC\right).$$

We deduce from that, using (H_1),

$$\sum_{q+j=m; q \geq k-\mu+\ell(\mu+\nu-1)} r_1^{-[q-(k-\mu+\ell(\mu+\nu-1))]}(2nC)^{j+k} M_{j+k}$$

$$\leq \sum_{q+j=m; q \geq k-\mu+\ell(\mu+\nu-1)} (nC)^{q-(k-\mu+\ell(\mu+\nu-1))} M_{q-(k-\mu+\ell(\mu+\nu-1))}$$

(10.7)

$$\times h_M^{-1}\left(r_1 nC\right)(2nC)^{j+k} M_{j+k}$$

$$\leq h_M^{-1}\left(r_1 nC\right)(2nC)^{m+\mu-\ell(\mu+\nu-1)} M_{m+\mu-\ell(\mu+\nu-1)} \sum_{t=0}^{\infty}\left(\frac{1}{2}\right)^t$$

$$\leq 2h_M^{-1}\left(r_1 nC\right)(2nC)^{m+\mu-\ell(\mu+\nu-1)} M_{m+\mu-\ell(\mu+\nu-1)}.$$

This gives the anounced result, because the number of terms of the sum $\sum_{|K| \leq \ell}$ in (10.1) is bounded by n^ℓ.

11. Lemma. *Let $\mathcal{B}$ belonging to $\mathcal{F}_n$ and α an integer, $\alpha > 0$. Write*

$$(11.1) \qquad \Phi^\alpha \mathcal{B} = \sum_{j \in \mathbb{N}} H_j(\Phi^\alpha \mathcal{B})(x)$$

and suppose that there exist $\beta \geq -1, C_{17}$ and C_{18} such that

$$(11.2) \qquad \begin{aligned} &H_j(\Phi^\alpha \mathcal{B}) \equiv 0 \ for \ j \leq \beta \leq \mu\alpha \ and \\ &\sup_{z \in \Omega_1} \mid H_j(\Phi^\alpha \mathcal{B})(z) \mid \leq C_{17} C_{18}^{j-\beta} M_{j-\beta} \ if \ j > \beta. \end{aligned}$$

Then, we have

$$\mathcal{B} = \sum_{j \in \mathbb{N}} H_j \mathcal{B}(x) \ \text{with}$$

(11.3)
$$\sup_{z \in \Omega_1} \mid H_j \mathcal{B}(z) \mid \leq 2C_{17} \left(\frac{C_3}{2} \right)^{-\alpha} h^{-1} \left(\frac{C_{18} r_1}{2} \right) (C_{18})^{\mu\alpha - \beta + j} M_{\mu\alpha - \beta + j}.$$

Proof. For z in Ω_1 and ζ in $\mathbb{C}, \mid \zeta \mid \leq r_1$, we can write

(11.4)
$$\Phi(\zeta z) = \zeta^\mu \Psi(\zeta, z).$$

Then, from (2.3), we get

(11.5)
$$\mid \Psi(\zeta, z) \mid \geq \frac{C_3}{2}.$$

Let

$$\Psi^{-\alpha}(\zeta, z) = \sum_{j=0}^{\infty} \psi_j(\alpha, z) \zeta^j,$$

It follows from the Cauchy formulas that

(11.6)
$$\mid \psi_j(\alpha, z) \mid \leq \left(\frac{C_3}{2} \right)^{-\alpha} r_1^{-j}.$$

We get then, for z in Ω_1,

(11.7)
$$\Phi^\alpha(\zeta z)\mathcal{B}(\zeta z) = \sum_{j \in \mathbb{N}} H_j(\Phi^\alpha \mathcal{B})(z)\zeta^j = \zeta^{\mu\alpha}\Psi^\alpha(\zeta, z) \sum_{j \in \mathbb{N}} H_j \mathcal{B}(z)\zeta^j$$

and

(11.8)
$$\zeta^{\mu\alpha} \sum_{j \in \mathbb{N}} H_j \mathcal{B}(z)\zeta^j = \sum_{j=0}^{\infty} \psi_j(\alpha, z)\zeta^j \sum_{j \geq \beta} H_j(\Phi^\alpha \mathcal{B})(z)\zeta^j$$
$$= \sum_{s \in \mathbb{N}} \zeta^s \sum_{t+j=s} \psi_t(\alpha, z) H_j(\Phi^\alpha \mathcal{B})(z).$$

We have

(11.9)
$$\sum_{t+j=s} \psi_t(\alpha, z) H_j(\Phi^\alpha \mathcal{B})(z) \equiv 0 \ \text{for} \ s \leq \beta, \ \text{and}$$
$$\left| \sum_{t+j=s} \psi_t(\alpha, z) H_j(\Phi^\alpha \mathcal{B})(z) \right| \leq \sum_{j+t=s} \left(\frac{C_3}{2} \right)^{-\alpha} r_1^{-t} C_{17} C_{18}^{j-\beta} M_{j-\beta}.$$

As in the proof of lemma 10, we remark that

(11.10)
$$r_1^{-t} \leq \left(\frac{C_{18}}{2} \right)^t M_t h_M^{-1} \left(\frac{r_1 C_{18}}{2} \right),$$

and hence we get

$$\left| \sum_{t+j=s} \psi_t(\alpha, z) H_j(\Phi^\alpha \mathcal{B})(z) \right|$$

(11.11)
$$\leq C_{17} \left(\frac{C_3}{2} \right)^{-\alpha} h_M^{-1} \left(\frac{r_1 C_{18}}{2} \right) \sum_{j+t=s} 2^{-t} C_{18}^{t+j-\beta} M_t M_{j-\beta}$$

$$\leq 2 C_{17} \left(\frac{C_3}{2} \right)^{-\alpha} h_M^{-1} \left(\frac{r_1 C_{18}}{2} \right) C_{18}^{s-\beta} M_{s-\beta}.$$

We have

$$H_i \mathcal{B}(z) = \sum_{t+j=i+\mu\alpha} \psi_t(\alpha, z) H_j(\Phi^\alpha \mathcal{B})(z),$$

and hence

$$\mid H_i \mathcal{B}(z) \mid \leq 2 C_{17} \left(\frac{C_3}{2} \right)^{-\alpha} h_M^{-1} \left(\frac{r_1 C_{18}}{2} \right) C_{18}^{i+\mu\alpha-\beta} M_{i+\mu\alpha-\beta}.$$

Let d be a strictly positive integer. We denote by $M^{(d)}$ the sequence defined by $M_n^{(d)} = M_{dn}$ for any n.

12. Proposition. *We have, for any $\mathcal{A}$ in $\mathcal{F}_n$ and any multi-index L of length ℓ,*

(12.1)
$$\sup_{|x|=1} \left| H_j \frac{\partial^\ell}{\partial z^L}(\mathcal{A}) \circ F(x) \right| \leq \ell! C_{19} (C_{20} C)^{j+(\mu-\nu+1)\ell} M_{j+(\mu-\nu+1)\ell} \parallel \mathcal{A} \circ F \parallel_{C,M}$$

and hence

(12.2)
$$\parallel \mathcal{A} \parallel_{C_{21} C^{\mu-\nu+1}, M^{(\mu-\nu+1)}} \leq C_{22} \parallel \mathcal{A} \circ F \parallel_{C,M} .$$

The constants C_{20} and C_{21} depend only on F. The constants C_{19} and C_{22} depend also on C.

Proof. Using (6.2) and the notation (10.1), we have

(12.3)
$$\Phi^{2\ell-1} \frac{\partial^\ell}{\partial u^L}(\mathcal{A}) \circ F = \mathcal{S}_L.$$

By (10.2), we have a control of $\sup_{z \in \Omega_1} \mid H_j \mathcal{S}_L(z) \mid$. From lemma 11 applied with $\alpha = 2\ell-1, \beta = \ell(\mu+\nu-1)-\mu$ and $\mathcal{B} = \frac{\partial^\ell}{\partial u^L}(\mathcal{A}) \circ F$ we get a control of $\sup_{z \in \Omega_1} \mid H_j \mathcal{B}(z) \mid$. From there, using (8.1), we deduce (12.1).

The inequality (12.2) is a consequence if we consider for any L the constant term in $\frac{\partial^\ell}{\partial u^L}(\mathcal{A}) \circ F$. This term is equal to $L! a_L$; we now use the fact that $\frac{\ell!}{L!}$ is bounded by n^ℓ to conclude.

We show in the next section that the results given here are the best one should expect. We discuss first the case when condition (1.4) is not satisfied.

13. Lemma [2]. *Let F be a holomorphic map from $\mathbb{C}^n$ to $\mathbb{C}^n$ defined in a ball $B(0, r_0)$ in $\mathbb{C}^n$ satisfying $F(0) = 0$ and whose Jacobian determinant Φ is identically zero. There exist two holomorphic maps F_1 and F_2 from $\mathbb{C}^n$ to $\mathbb{C}^n$ defined in a neighborhood of 0 satisfying $F_1(0) = F_2(0) = 0$, whose Jacobian determinants are*

not identically zero in a neighborhood of 0 *such that* $F_1 \circ F \circ F_2$ *depends only on the* $n-1$ *first variables.*

Write

$$(13.1) \qquad\qquad G = F_1 \circ F \circ F_2.$$

14. Lemma [2]. *For any large enough integer* α, *there exists a polynomial* P_α *of* n *variables satisfying the following properties:*

$$(14.1) \qquad\qquad P_\alpha(z) = \sum_{K,k\leq\alpha} a_{K,\alpha} z^K,$$

$$(14.2) \qquad\qquad \sup_{K,k\leq\alpha} |a_{K,\alpha}| = 1,$$

$$(14.3) \qquad\qquad \mathrm{ord}(P_\alpha \circ G) \geq C_{23}\alpha^{n/n-1}.$$

The constant C_{23} *depends only on* G *defined in (13.1).*

15. Proposition. *Let* F *be a holomorphic map from* $\mathbb{C}^n$ *to* $\mathbb{C}^n$ *defined in a ball* $B(0, r_0)$ *in* $\mathbb{C}^n$ *satisfying* $F(0) = 0$ *and whose Jacobian determinant* Φ *is identically zero. Let* M *be a sequence of strictly positive real numbers satisfying* (H_1) *and* (H_2). *Let* C *and* D *two strictly positive constants and* e *an integer,* $e \geq 1$. *There exists a formal power series* $\mathcal{A}$ *which does not satisfy the inequality*

$$\|\mathcal{A}\|_{D,M^{(e)}} \leq D\|\mathcal{A}\circ F\|_{C,M} < \infty.$$

Proof. We use lemmas 13 and 14 and the function G defined in (13.1).

From (14.1) and (14.2), we have

$$(15.1) \qquad\qquad \|P_\alpha\|_{D,M^{(e)}} \geq \frac{1}{D^\alpha M_{e\alpha}}, \ \text{ for any } \alpha \text{ large enough.}$$

From (1.1), (1.2) and (14.2), we get, for any multi-index L and any α large enough,

$$(15.2) \qquad\qquad \left| \frac{\partial^\ell}{\partial z^L}(P_\alpha \circ G)(0) \right| \leq \ell! C_{24}^{\ell+1}.$$

The constant C_{24} depends only on G.

Denote by $E(\alpha)$ the integer part of $C_{23}\alpha^{n/n-1} + 1$. From (14.3) and (15.2), we have, for α large enough,

$$(15.3) \qquad\qquad \|P_\alpha \circ G\|_{C,M} \leq \frac{C_{25}^{E(\alpha)}}{M_{E(\alpha)}}.$$

The constant C_{25} depends only on C and G.

We get from (15.1), (15.3), (H_2) and the previous definition of $E(\alpha)$ that there exists α large enough such that

$$\|P_\alpha\|_{D,M^{(e)}} \leq D \, \|P_\alpha \circ G\|_{C,M}$$

is not fulfilled.

Since, by construction, the maps F_1 and F_2 satisfy inequalities (12.2), we get from there the announced result.

16. Theorem C. *Let F be a holomorphic map from $\mathbb{C}^n$ to $\mathbb{C}^n$ defined in a neigh-borhood of 0 with Jacobian determinant Φ.*

Let M be a sequence of strictly positive real numbers satisfying (H_1) and (H_2). Let C be a strictly positive constant. The two following properties are equivalent:

(C.1) $$\operatorname{ord}(\Phi) < \infty,$$

(C.2) $\quad$ *there exist $D, D > 0$ and $d \in \mathbb{N}$, $d \geq 1$ such that for any $\mathcal{A} \in \mathcal{F}_n$, we have $\|\mathcal{A}\|_{D,M^{(d)}} \leq D\|\mathcal{A} \circ F\|_{C,M}$.*

Proof. This is a consequence of propositions 12 and 15.

We can give now a precise version of "(A.1) implies (A.2)".

17. Theorem. *Let F be a holomorphic map from $\mathbb{C}^n$ to $\mathbb{C}^n$ defined in a neigbor-hood of 0 and let Φ be its Jacobian determinant. Let μ denote the vanishing order of Φ and ν defined as in section 3.*

Suppose that

(A.1) $$\operatorname{ord}(\Phi) < \infty.$$

Then

(A.2) $\quad$ *for any $\mathcal{A} \in \mathcal{F}_n$, $\operatorname{ord}(\mathcal{A} \circ F) \leq (\mu - \nu + 1)\operatorname{ord}(\mathcal{A})$.*

Proof. Denote by o_1 the vanishing order of $\mathcal{A} \circ F$ and by o_2 the vanishing order of $\mathcal{A}$.

From definition (10.1)

$$\mathcal{S}_L = \sum_{K \in \mathbb{N}^n; |K| \leq \ell} T_{K,L} \frac{\partial^k}{\partial z^K}(\mathcal{A} \circ F),$$

and by (4.1) it is clear that

(17.1) $$\operatorname{ord}(\mathcal{S}_L) \geq o_1 - \mu + \ell(\mu + \nu - 1).$$

On the other hand, there exists a multi-index P of length $p = o_2$ such that

$$\frac{\partial^p}{\partial u^P}(\mathcal{A}) \neq 0.$$

By (12.3), we get

(17.2) $$\operatorname{ord}(\mathcal{S}_P) = (2o_2 - 1)\mu.$$

Using (17.1) with $L = P$ and (17.2), we have

$$(2o_2 - 1)\mu \geq o_1 - \mu + o_2(\mu + \nu - 1).$$

This gives the result.

18. Remarks. **(a)** The estimate (12.2) is the best possible, as the following example shows.

Example. Let $\sigma_1, \cdots, \sigma_n$ be the elementary symmetric functions in n variables. Denote by S the map from $\mathbb{C}^n$ to $\mathbb{C}^n$ defined by

$$S(x) = (\sigma_1(x), \sigma_2(x), ..., \sigma_n(x)).$$

It is easily seen that $\mu = \frac{n(n-1)}{2}$, and $\nu = \frac{(n-1)(n-2)}{2}$ and $\mu - \nu + 1 = n$.

If $\mathcal{A} = \sum_p a_p x_n^p$ is a formal power series depending only on the last variable x_n, we have

$$\|\mathcal{A}\|_{C^n, M^{(n)}} = \|\mathcal{A} \circ S\|_{C, M}.$$

Hence, inequality (C.2) is the best one should expect with $d = \mu - \nu + 1 = n$.

More generally, let $P_1, \cdots, P_n$ be homogeneous polynomials in n variables of degree respectively $d_1 \leq \cdots \leq d_n$. Denote by P the map from $\mathbb{C}^n$ to $\mathbb{C}^n$ defined by

$$P(x) = (P_1(x), P_2(x), ..., P_n(x)).$$

If the order of the Jacobian determinant of P is finite, one can easily prove that inequality (C.2) is optimal with $d = d_n$.

(b) Under the hypothesis $\mathrm{ord}(\Phi) = \infty$, a formal power series $\mathcal{A}$ can be constructed in such a way that, for any constant D and any integer d,

$$\|\mathcal{A}\|_{D, M^{(d)}} = \infty \quad \text{and} \quad \|\mathcal{A} \circ F\|_{1, M} < \infty.$$

(c) With slight modifications we can adapt the proof of proposition 12 when $M_p = 1$ for every p. Write $M = \mathbb{1}$ in that case. We get hence a new proof of

$$(\text{B.1}) \text{ implies } (\text{B.2}).$$

Moreover, if we assume that C has been chosen such that $2nC \geq \frac{1}{r_1}$, we have

$$\text{for any } \mathcal{A} \in \mathcal{F}_n, \|\mathcal{A}\|_{EC^{\mu-\nu+1}, \mathbb{1}} \leq E\|\mathcal{A} \circ F\|_{C, \mathbb{1}}.$$

This means that if the radius of convergence $\rho_{\mathcal{A} \circ F}$ is small enough, then the radius of convergence $\rho_{\mathcal{A}}$ satisfies the following inequality:

$$(\rho_{\mathcal{A} \circ F})^{\mu-\nu+1} \leq E\rho_{\mathcal{A}}.$$

Here, the radius of convergence of a power series $\mathcal{A}$ is the largest real number ρ such that $\mathcal{A}$ converges in the polydisk of center 0 and radius ρ.

The example given in (a) shows that the above inequality is the best possible. On this topic, see also J. C. Tougeron [8].

(d) We have shown that $d = \mu + \nu - 1$ is optimal for S in (a). Nevertheless, given a function F, finding the best possible integer d in (C.2) and the best constant D_1 in (A.2) would be of interest.

(e) Theorem C can be seen as a version of the Glaeser composite functions theorem "above a point" for ultradifferentiable classes ([4], [6]).

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U.M.R. C.N.R.S. 8628, Université Paris-Sud, Mathématiques - Bât. 425, 91405 Orsay Cedex, France
  E-mail address: `jacques.chaumat@math.u-psud.fr`

U.M.R. C.N.R.S. 8524, Université de Lille, U.F.R. de Mathématique,59655 Villeneuve D'Ascq Cedex, France
  E-mail address: `chollet@aglae.univ-lille1.fr`